Abstract

Here is what I did on Exercise 3(a) in our first Linear Algebra exam. The mathematical induction part seems rather simple and trivial, but I guess it really proves that D^k equals what it does.

We are told to consider an arbitrary $n \times n$ diagonal matrix:

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

Part(a) asks,*"What is D^k, where k is an arbitrary positive integer?"*.

Well, to get an idea what D^k is, I played around with it some, trying it with $k = 2, k = 3$, and $k = 4$. These were easy to do. For D^2, we know that $D^2 = DD$, so we have:

$$D^2 = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

This gives us:

$$D^2 = \begin{pmatrix} (d_{11})^2 & 0 & \cdots & 0 \\ 0 & (d_{22})^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^2 \end{pmatrix}$$

Similarly proceeding with D^3 and D^4, we get:

$$D^3 = \begin{pmatrix} (d_{11})^3 & 0 & \cdots & 0 \\ 0 & (d_{22})^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^3 \end{pmatrix}$$

and

$$D^4 = \begin{pmatrix} (d_{11})^4 & 0 & \cdots & 0 \\ 0 & (d_{22})^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^4 \end{pmatrix}$$

From these few examples, it seems like the general pattern for D^k is:

$$D^k = \begin{pmatrix} (d_{11})^k & 0 & \cdots & 0 \\ 0 & (d_{22})^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^k \end{pmatrix}$$

And, from just looking at D, it looks like this is what D^k equals. But here is some mathematical induction to prove that.

I want to prove that

$$D^n = \begin{pmatrix} (d_{11})^n & 0 & \cdots & 0 \\ 0 & (d_{22})^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^n \end{pmatrix}$$

is true for all n, where n is a positive integer. It is obvious that this is true for n=1, since we are given that

$$D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$$

Now I will assume that that the statement is true for n=k.

$$D^k = \begin{pmatrix} (d_{11})^k & 0 & \cdots & 0 \\ 0 & (d_{22})^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^k \end{pmatrix}$$

I've got to show that if this is true for n=k, then it is also true for n=k+1. So,

$$\begin{aligned} D^{k+1} &= D^k D \\ &= \begin{pmatrix} (d_{11})^k & 0 & \cdots & 0 \\ 0 & (d_{22})^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^k \end{pmatrix} \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \\ &= \begin{pmatrix} (d_{11})^k d_{11} & 0 & \cdots & 0 \\ 0 & (d_{22})^k d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^k d_{nn} \end{pmatrix} \\ &= \begin{pmatrix} (d_{11})^{k+1} & 0 & \cdots & 0 \\ 0 & (d_{22})^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^{k+1} \end{pmatrix} \end{aligned}$$

We now have what D^{k+1} equals. The statement is true for n=k+1! It is shown that if the statement is true for one number n=k, where n is any integer, then it is true for the next number n=k+1. And we know that it is true for n=1.

Therefore,

$$D^n = \begin{pmatrix} (d_{11})^n & 0 & \cdots & 0 \\ 0 & (d_{22})^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (d_{nn})^n \end{pmatrix}$$

for all n.

THE END!