I am not sure what you are trying to do. I think what you're trying to do is to prove that if A is 3×3 and $\text{col}_3 = \text{col}_1 + \text{col}_2$ then A has no inverse?

Am I correct in my thinking? Is this what you are trying to do?

If so, there are easier ways to think about the problem without bringing in the transpose. For example, if A is 3×3 and invertible, then the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution, namely $\mathbf{x} = \mathbf{0}$.

This is easily shown to be true. Suppose that A is invertible and $A\mathbf{x} = \mathbf{0}$. Then,

$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{0} \\ (A^{-1}A)\mathbf{x} &= \mathbf{0} \\ I\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0}. \end{aligned}$$

Therefore, if A is invertible, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ represent the columns of matrix A. Because $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, then

$$\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}.$$

This means that

$$\begin{aligned} 1\mathbf{a}_1 + 1\mathbf{a}_2 - 1\mathbf{a}_3 &= \mathbf{0} \\ [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} &= \mathbf{0} \\ A \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} &= \mathbf{0}. \end{aligned}$$

This nonzero solution of $A\mathbf{x} = \mathbf{0}$ informs us that A is not invertible (if it were invertible, there would be only the zero soluton).