All,

Greg Brown is taking a proof course at the University of California at San Diego. This is a sophomore level course that is required before taking any upper division math courses. He sent this question.

Prove: Let $f : A \to B$ be a function. Then f maps A onto B iff if whenever C is a set and $g : B \to C$ and $h : B \to C$ are functions such that $g \circ f = h \circ f$, it follows that $g = h$.

Proof: Suppose f maps A onto B, $g : B \to C$, $h : B \to C$, and $g \circ f = h \circ f$. To prove $g = h$, we must show Domain(g) = Domain(h) and if $x \in$ Domain(g), then $g(x) = h(x)$. Domain(g) = B = Domain(h). Let $x \in B$. Since f maps A onto B, $\exists a \in A$ such that $f(a) = x$. Then since $g \circ f = h \circ f$, $g \circ f(a) = h \circ f(a)$, and thus $g(x) = f(x)$. But how does one prove the converse?

My response:

Greg, the essential idea is to grasp the meaning of $g \circ f = h \circ f$. You can think of this as meaning that

$$g(f(x)) = h(f(x))$$

for all x in A, or better yet, you can think of it as meaning

$$g(y) = h(y)$$

for all y in $f(A)$, the image of A under f. Essentially, this means that g and h agree on the set $f(A)$ (see Figure 1). The function $g \circ f$ is sometimes called the *restriction* of g to $f(A)$. Similarly,

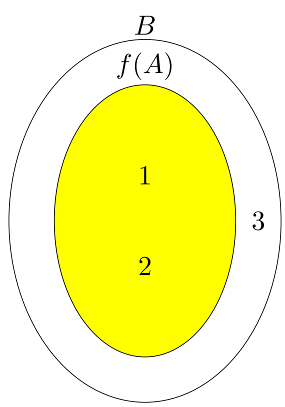

Figure 1: The image of A under f is contained in B.

the function $h \circ f$ is the restriction of h to $f(A)$.

Thus, when we say that $(g \circ f)(x) = (h \circ f)(x)$ for all x in A, we are saying that the functions g and h agree on the set $f(A)$.

Now, if f is not onto B, then there is an object in B that is not in the range of f. That is, there is an object in B that is not in $f(A)$, as shown in Figure 1. Both 1 and 2 are in $f(A)$, but 3 is not. Therefore, the function f is not onto B.

Let $C = \{8, 9, 10, 11\}$. Now, in this case we can define two functions $g : B \to C$ and $h : B \to C$ that agree on $f(A)$, but they do not agree on all of B. For example, let $g : B \to C$ be defined by

$$\begin{array}{ccc} & g & \\ 1 & \longrightarrow & 9 \\ 2 & \longrightarrow & 10 \\ 3 & \longrightarrow & 11 \end{array}$$

Next, define $h : B \to C$ by

$$\begin{array}{ccc} & h & \\ 1 & \longrightarrow & 9 \\ 2 & \longrightarrow & 10 \\ 3 & \longrightarrow & 8 \end{array}$$

Note that g and h agree on $f(A) = \{1, 2\}$, but g and h do not agree on all of $B = \{1, 2, 3\}$, as required by the hypothesis. This is your contradiction.

Hope this helps. By the way, this problem had me pretty good and stumped for most of the day. As we worked in the lab today, I just let it mill around in my head while I concentrated on other things. Finally, just about midnight a solution came to me. Magic!