On Wednesday, Jonathan Gent approached me with a question after class. I had started with vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix},$$

and proceeded to find

$$\mathbf{A} = \mathbf{a},$$

and

$$\mathbf{B} = \mathbf{a} - \frac{\mathbf{b} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}.$$

On the third step, I drew a diagram projecting $\mathbf{c}$ onto the plane spanned by the vectors $\mathbf{A}$ and $\mathbf{B}$. The Gram-Schmidt formula at this third step provided

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A} - \frac{\mathbf{c} \cdot \mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} \mathbf{B}.$$

Now, Jonathan then proposed the following question. If $\mathbf{C} = \mathbf{c} - \mathbf{p}$, can't we find the vector $\mathbf{p}$ using the the matrix that projects onto the space spanned by $\mathbf{A}$ and $\mathbf{B}$?

I scratched out a few notes, then got stuck, but Jonathan returned later in the day with a completed solution. Here is Jonathan's proof.

Proof: Let

$$M = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix},$$

where $\mathbf{A}$ and $\mathbf{B}$ are as defined above. Then, the projection matrix onto the space spanned by $\mathbf{A}$ and $\mathbf{B}$ is

$$P = M(M^T M)^{-1} M^T.$$

Now,

$$\begin{aligned} \mathbf{C} &= \mathbf{c} - \mathbf{p} \\ &= \mathbf{c} - P\mathbf{c} \\ &= \mathbf{c} - M(M^T M)^{-1} M^T \mathbf{c}, \end{aligned}$$

However,

$$M^T M = \begin{pmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^T & 0 \\ 0 & \mathbf{B}^T \mathbf{B} \end{pmatrix} = \begin{pmatrix} |\mathbf{A}|^2 & 0 \\ 0 & |\mathbf{B}|^2 \end{pmatrix}.$$

Therefore,

$$(M^T M)^{-1} = \begin{pmatrix} \dfrac{1}{|\mathbf{A}|^2} & 0 \\ 0 & \dfrac{1}{|\mathbf{B}|^2} \end{pmatrix}.$$

Thus,

$$\begin{aligned} P &= M(M^T M)^{-1} M^T \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \dfrac{1}{|\mathbf{A}|^2} & 0 \\ 0 & \dfrac{1}{|\mathbf{B}|^2} \end{pmatrix} \begin{pmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\mathbf{A}}{|\mathbf{A}|^2} & \dfrac{\mathbf{B}}{|\mathbf{B}|^2} \end{pmatrix} \begin{pmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{pmatrix} \\ &= \left(\frac{\mathbf{A}\mathbf{A}^T}{|\mathbf{A}|^2} + \frac{\mathbf{B}\mathbf{B}^T}{|\mathbf{B}|^2} \right). \end{aligned}$$

Finally,

$$\begin{aligned}
\mathbf{C} &= \mathbf{c} - P\mathbf{c} \\
&= \mathbf{c} - M(M^T M)^{-1} M^T \mathbf{c} \\
&= \mathbf{c} - \left(\frac{\mathbf{A}\mathbf{A}^T}{|\mathbf{A}|^2} + \frac{\mathbf{B}\mathbf{B}^T}{|\mathbf{B}|^2} \right) \mathbf{c} \\
&= \mathbf{c} - \left(\frac{\mathbf{A}\mathbf{A}^T \mathbf{c}}{|\mathbf{A}|^2} + \frac{\mathbf{B}\mathbf{B}^T \mathbf{c}}{|\mathbf{B}|^2} \right) \\
&= \mathbf{c} - \left(\frac{\mathbf{A}(\mathbf{A} \cdot \mathbf{c})}{|\mathbf{A}|^2} + \frac{\mathbf{B}(\mathbf{B} \cdot \mathbf{c})}{|\mathbf{B}|^2} \right) \\
&= \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{A}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A} - \frac{\mathbf{c} \cdot \mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} \mathbf{B}.
\end{aligned}$$

So, yep, they are the same. Nice going, Jonathan.