Actually, the subject is deep, but I'll try to give the short answer (if there is one). Suppose you have a linear transformation T that maps the vector space V to the vector space W ($T : V \to W$).

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a basis for V. There is a natural transformation $[\ ]_{\mathcal{B}} : V \to \mathbb{R}^n$ defined by

$$[\ ]_{\mathcal{B}} : \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n \to \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

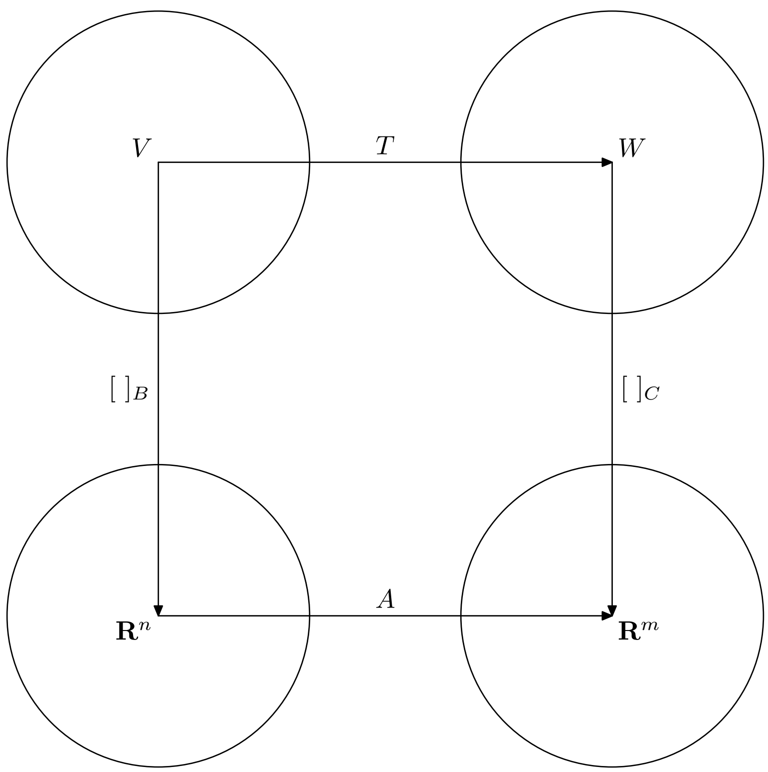
It is easy to show that $[\ ]_{\mathcal{B}}$ is a linear transformation, that it is one-to-one, and onto $\mathbb{R}^n$. Thus, $[\ ]_{\mathcal{B}}$ is an isomorphism and the vector space V is isomorphic to $\mathbb{R}^n$.

In a similar manner, let $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_,\}$ be a basis for the vector space W. The coordinate transformation

$$[\ ]_{\mathcal{C}} : \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \cdots + \beta_m\mathbf{w}_m \to \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

maps W isomorphically to $\mathbb{R}^m$.

Now, the linear transformation T maps V to W, but the matrix A of transformation maps $\mathbb{R}^n$ to $\mathbb{R}^m$.



The matrix of transformation A is completely determined by its action on the basis elements. First, write $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, $\ldots$, $T(\mathbf{v}_n)$ as linear combinations of the basis elements for W.

$$\begin{aligned}
T(\mathbf{v}_1) &= \alpha_{11}\mathbf{w}_1 + \alpha_{21}\mathbf{w}_2 + \cdots + \alpha_{m1}\mathbf{w}_m \\
T(\mathbf{v}_2) &= \alpha_{12}\mathbf{w}_1 + \alpha_{22}\mathbf{w}_2 + \cdots + \alpha_{m2}\mathbf{w}_m \\
&\vdots \\
T(\mathbf{v}_n) &= \alpha_{1n}\mathbf{w}_1 + \alpha_{2n}\mathbf{w}_2 + \cdots + \alpha_{mn}\mathbf{w}_m
\end{aligned}$$

You can now form the matrix of transformation A by placing these coefficients down *each column* of the matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$