Lee asked about the mystery of the multipliers winding up in the positions they do in the lower triangular matrix L in the factorization $A = LU$. Consider, if you will,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ -3 & 4 & 2 \end{pmatrix}.$$

The first multiplier $l_{21} = A(2,1)/A(1,1) = 2/1 = 2$. Consequently, the first elementary row operation to apply is to subtract 2 times row 1 from row 2.

$$E_{21}A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ -3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ -3 & 4 & 2 \end{pmatrix}$$

We store this multiplier $l_{21} = 2$ for posterity in the lower triangular matrix L.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second multiplier is $l_{31} = A(3,1)/A(1,1) = -3/1 = -3$. Consequently, the second elementary row operation to apply is to subtract -3 times row 1 from row 3.

$$E_{31}(E_{21}A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ -3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 10 & 11 \end{pmatrix}$$

We store this multiplier $L_{31} = -3$ for posterity in the lower triangular matrix L.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

The third and final multiplier is $l_{32} = 10/-5 = -2$. Consequently, the third and final elementary row operation is to subtract -2 times row 2 from row 3.

$$E_{32}(E_{31}E_{21}A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 10 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 1 \end{pmatrix} = U$$

We store this multiplier $l_{32} = -2$ for posterity in the lower triangular matrix L.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix}$$

Thus, in symbols, the row reduction is represented as

$$E_{32}E_{31}E_{21}A = U.$$

If we multiply both sides of this equation *on the left* by $(E_{32}E_{31}E_{21})^{-1}$, we have

$$A = (E_{32}E_{31}E_{21})^{-1}U.$$

However, the inverse of a product is the product of the inverses, but in reverse order. Thus, we can write

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U.$$

Thus,

$$A = LU,$$

where

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}.$$

Now, there is nothing easier than inverting an elementary matrix. To "undo" subtracting 2 times row 1 from row 2, simply subtract -2 times row 1 from row 2. Thus,

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad E_{21}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, to "undo" subtracting -3 times row 1 from row 3, simply subtract 3 times row 1 from row 3. Thus,

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad E_{31}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.$$

Finally, to "undo" subtracting -2 times row 2 from row 3, simply subtract 2 times row 2 from row 3. Thus,

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Now to the crux of Lee's question. The product of lower triangular matrices is also a lower triangular matrix. However, note what happens when we multiply the elementary matrices in the order $E_{32}E_{31}E_{21}$.

$$\begin{aligned} E_{32}E_{31}E_{21} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \end{aligned} \tag{1}$$

Note that the opposite of each multiplier appears in the $(2,1)$ and $(3,2)$ positions, but the entry in the $(3,1)$ position is strange! If you look carefully, you will see that there are *two* row operations that affect the entry in the $(3,1)$ position.

Similarly, first note that the inverse of a lower triangular matrix is also lower triangular. Next, note what happens when we multiply the inverses of our elementary matrices in the following order.

$$\begin{aligned} L &= E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix} \end{aligned} \tag{2}$$

Note that the multipliers $l_{21} = 2$, $l_{31} = -3$, and $l_{32} = -2$ appear in $L(2,1)$, $L(3,1)$, and $L(3,2)$ postions as they should. Also, unlike the action in (1), there is only *one* row operation that affects the entry in the $(3,1)$ position.

I hope this helps.