Let

$$\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$$

be the standard basis for R^2.

Now, suppose that I know what the linear transformation T does to each of these basis elements. Since $\mathcal{B}$ is a basis, any element in R^2 can be written as a linear combination of the vectors in $\mathcal{B}$. That is, if

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2,$$

then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \\ &= T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) \\ &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) \\ &= \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= A\mathbf{x}. \end{aligned}$$

Therefore, all one need do is calculate what T does to each basis element, putting the responses in the columns of the matrix of transformation.