Strang, Page 105, Exercise #7

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When you get a "true" or "false" question in Strang, it is not enough to simply answer "true" or "false" and move on to the next question. On the contrary, here is how you should proceed.

1. If you think the item is false, provide a counterexample that clearly demonstrates its falsehood.
2. If you think the item is true, prove it for the general case.

For example, consider the statement in Exercise #7a:

> "The block matrix
> $$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$$
> is automatically symmetric."

The approach here should be to play a bit. Try it for some matrices to see if it rings true. For example, suppose that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix}.$$

Clearly, this result is not symmetric across its main diagonal. Hence, the statement is false.

On the other hand, consider the "statement" in Exercise #7c.

> "If A is not symmetric, then A^{-1} is not symmetric."

After some experimentation, I could not come up with a counterexample. This leads me to think that the statement is true. It is now my duty to provide a proof of the statement.

Prove. If A is not symmetric, then A^{-1} is not symmetric.

Proof. Assume that A is not symmetric. Now assume that A^{-1} **is** symmetric for purposes of contradiction. If A^{-1} is symmetric, then

$$(A^{-1})^T = A^{-1}.$$

We know that the transpose of the inverse is the inverse of the transpose, so we can write

$$(A^T)^{-1} = A^{-1}.$$

We can take the inverse of both sides of this equation.

$$\left[(A^T)^{-1}\right]^{-1} = \left[A^{-1}\right]^{-1}.$$

The inverse of the inverse takes us back to the original matrix and we can write

$$A^T = A.$$

This last statement implies that A is symmetric, which contradict the hypothesis that A was not symmetric.

Hence, my assumption that A^{-1} was symmetric is incorrect (it led to a contradiction). Hence, A^{-1} is **not** symmetric and my proof is complete.

This is exactly the type of approach that should be taken on any homework that involves true-false type questions.