

Suppose that matrix  $A$  is  $m \times n$  and matrix  $B$  is  $n \times p$ . Then the matrix multiplication is possible. Thinking in blocks,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p].$$

However, each  $A\mathbf{b}_j$  is a linear combination of the columns of matrix  $A$ .

$$\begin{aligned} A\mathbf{b}_j &= [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n] \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \\ &= b_{1j}\mathbf{a}_1 + b_{2j}\mathbf{a}_2 + \dots + b_{nj}\mathbf{a}_n \end{aligned}$$

Therefore, each column of matrix  $AB$  is a linear combination of the columns of  $A$ . This means that everything in the column space of  $AB$  also belongs to the column space of  $A$ .

For example, if

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \left( \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \left( 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \end{aligned}$$

Note that each column of  $AB$  is a linear combination of the columns of  $A$ . Thus, any combination of the columns of  $AB$  will actually be a combination of the columns of  $A$ . Indeed, only the first two columns of matrix  $A$  are used, so the column space of  $AB$  is properly contained in the column space of matrix  $A$ .