Let's craft a random 4×6 matrix with integer entries between -5 and 5.

```
A=randint(4,6,5,3)
```

```
A =

    -3          0          3         -2          2         -2
     2          0         -2          1         -1          1
     2          2         -4         -2          2          2
     0          0          0          2         -2          2
```

Set up the matrix equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} -3 & 0 & 3 & -2 & 2 & -2 \\ 2 & 0 & -2 & 1 & -1 & 1 \\ 2 & 2 & -4 & -2 & 2 & 2 \\ 0 & 0 & 0 & 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Put the matrix A in reduced row echelon form.

```
R=rref(A)
```

```
R =

     1          0         -1          0          0          0
     0          1         -1          0          0          2
     0          0          0          1         -1          1
     0          0          0          0          0          0
```

Identify the pivot columns. There are pivots in column 1, column 2, and column 4. Therefore, x_1, x_2 and x_4 are *pivot variables*. Of course, this makes x_3, x_5, and x_6 *free variables*.

The matrix R represents the system of equations

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 + 2x_6 &= 0 \\ x_4 - x_5 + x_6 &= 0. \end{aligned}$$

Solve each equation for its pivot variable.

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 - 2x_6 \\ x_4 &= x_5 - x_6 \end{aligned}$$

Therefore, our solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 - 2x_6 \\ x_3 \\ x_5 - x_6 \\ x_5 \\ x_6 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

Therefore, any vector in the nullspace of matrix A is a linear combination of the "special vectors"

$$\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\\1\\0\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\-2\\0\\-1\\0\\1\end{pmatrix}.$$

Is there a faster way to pick off these "special" vectors? You betcha! Consider the reduced row echelon form of A again.

$$R = \begin{pmatrix}1 & 0 & -1 & 0 & 0 & 0\\0 & 1 & -1 & 0 & 0 & 2\\0 & 0 & 0 & 1 & -1 & 1\\0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

We seek vectors $\mathbf{x}$ such that $R\mathbf{x} = \mathbf{0}$. Look at column three, the first *free* column. It can be written as a linear combination of the pivot columns that proceed it. Notice that

$$1\text{col}1 + 1\text{col}2 + 1\text{col}3 = \mathbf{0}.$$

Of course, this also means that

$$1\text{col}1 + 1\text{col}2 + 1\text{col}3 + 0\text{col}4 + 0\text{col}5 + 0\text{col}6 = \mathbf{0}.$$

Therefore,

$$\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}$$

is in the nullspace of matrix R. Because solutions of $R\mathbf{x} = \mathbf{0}$ are also solutions of $A\mathbf{x} = \mathbf{0}$, the vector $(1, 1, 1, 0, 0, 0)^T$ is also in the nullspace of matrix A.

Now, look at column at column 5 of matrix R. It is the very next *free* column. It can be written as a linear combination of the pivot columns that proceed it. Indeed,

$$1\text{col}4 + 1\text{col}5 = \mathbf{0}.$$

Of course, this also means that

$$0\text{col}1 + 0\text{col}2 + 0\text{col}3 + 1\text{col}4 + 1\text{col}5 + 0\text{col}6 = \mathbf{0}.$$

Therefore,

$$\begin{pmatrix}0\\0\\0\\1\\1\\0\end{pmatrix}$$

is a solution of $R\mathbf{x} = \mathbf{0}$ and an element of the nullspace of R. For the same reasons mentioned above, it is also an element of the nullspace of A.

Finally, look at the sixth column of matrix R, the last free column. It can be written as a linear combination of the pivot columns that proceed it. Indeed

$$-2\text{col}2 - 1\text{col}4 + 1\text{col}6 = \mathbf{0}.$$

Of course, this also means that

$$0\text{col1} - 2\text{col2} + 0\text{col3} - 1\text{col4} + 0\text{col5} + 1\text{col6} = \mathbf{0}.$$

Therefore,

$$\begin{pmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

is a solution of $R\mathbf{x} = \mathbf{0}$ and is in the column space of R. Of course, this also puts $(0, -2, 0, -1, 0, 1)^T$ in the column space of matrix A.

Lee, this is not guessing. Rather, this ease in picking off the "special" vectors in the nullspace from the reduced row echelon form R is essential to the understanding of the nullspace. It is the "correct" way to think and work.

Here's what I suggest. Download `randint.m` from

`http://online.redwoods.cc.ca.us/instruct/darnold/linalg/atlast.zip`

Generate a random 3×7 matrix with integer entries between -10 and 10 with rank 2.

```
 A=randint(3,7,10,2)

A =

    -7     7    -7    -1    -3    -3     1
    -1     1    -1     9    -5    -5    -9
    -5     5    -5    -5     0     0     5
```

Reduce.

```
 R=rref(A)

R =

    1.0000   -1.0000    1.0000         0    0.5000    0.5000         0
         0         0         0    1.0000   -0.5000   -0.5000   -1.0000
         0         0         0         0         0         0         0
```

Generate the "special" nullspace vectors using the technique of equation (1). Next, pick off the "special" nullspace vectors direct from the reduced form of the matrix with no work, as we did above. Make sure the two techniques match.

Now, generate another matrix of different dimension and rank. Redcue. List the "special" vectors in the nullspace.

Again. Generate another matrix of different dimension and rank. Reduce. List the "special" vectors in the nullspace.

Over and over again until it becomes second nature, this plucking of "special" nullspace vectors from the reduced form of the matrix.

Now, go back to the exercises and see if they are easier.