Strang
Exercise 26, Page 133

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Well, so that you have a chance to play, think about 2×2 matrices A.

1. If $\text{rank}(A) = 2$, then the dimension of $C(A)$ is 2 and $N(A)$ will have dimension 0. One space is Z and the other is $\mathbb{R}^2$. They cannot be equal.
2. If $\text{rank}(A) = 0$, then the dimension of $C(A)$ is 0 and $N(A)$ will have dimension 2. One space is Z and the other is $\mathbb{R}^2$. They cannot be equal.
3. Therefore, A must have rank 1. This means that one column of A is a multiple of the other.

Now, not any rank 1 matrix will suffice. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

then the column space is all multiples of $[1, 2]^T$ and the nullspace is all multiples of $[-2, 1]^T$. These are both lines passing through the origin, but with different slopes. They cannot be the same.

Now, we know matrix A must look like the following.

$$A = \begin{bmatrix} a & ka \\ b & kb \end{bmatrix}$$

The column space of A will be all multiples of $[a, b]^T$. The nullspace of A will be all multiples of $[k, -1]^T$. How can we get these lines to point in the same direction? Of course! It must be that $[k, -1]^T$ is a multiple of $[a, b]^T$. That is,

$$\begin{bmatrix} k \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}.$$

This means that

1. $k = \lambda a$, and
2. $-1 = \lambda b$.

Solving the second equation,

$$\lambda = -\frac{1}{b}.$$

Subbing this result in the first equation,

$$k = -\frac{a}{b}.$$

Thus,

$$A = \begin{bmatrix} a & ka \\ b & kb \end{bmatrix},$$

with $k = -a/b$, becomes

$$A = \begin{bmatrix} a & -a^2/b \\ b & -a \end{bmatrix}.$$

For example, with $a = 1$ and $b = 2$,

$$A = \begin{bmatrix} 1 & -1/2 \\ 2 & -1 \end{bmatrix}.$$

I'll leave it to you to check that $C(A) = N(A)$.