We are assuming that we are working with a $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let r be the rank (number of pivot columns) of matrix A.

I will look at part (a) of this question. I am asked to find all know relations between r, m, and n if $A\mathbf{x} = \mathbf{b}$ has no solution for some $\mathbf{b}$.

- If $m < n$, that is, if the number of rows is fewer than the number of columns, then the reduced form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ will exhibit free variables. In this case, the equation $A\mathbf{x} = \mathbf{b}$ always has solutions. Therefore, in this situation (no solution for some $\mathbf{b}$, it is not the case that $m < n$. That is, m must be greater than or equal to n.

- Thus, there are two cases, $m = n$ and $m > n$, that I must consider.

  1. Consider the case where $m = n$. If the number of pivot columns equals the number of columns, that is, if $r = n$, then the equation $A\mathbf{x} = \mathbf{b}$ always has a unique solution. So, it must be the case that $r < n$. As an example, consider $A\mathbf{x} = \mathbf{b}$, where

     $$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

     In this case, there is no solution. But in the case where

     $$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

     there are many solutions. In summary, one possibility for this case is $m = n$ and $r < n$.

  2. Consider the case where $m > n$, that is, the number of rows is greater than the number of columns of matrix A. In this case, even if $r = n$, it is possible that there is no solution for some $\mathbf{b}$. As an example, consider

     $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

     But, even if $r < n$, the number of pivots is strictly less than the number of columns of A, it is still possible that there are no solutions for some $\mathbf{b}$. As an example, consider

     $$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

- Thus, in summary, if $m \leq n$ and $r \leq n$, it is possible that the system has no solution for some $\mathbf{b}$.