In class today, I botched problem 34 on page 140 pretty badly. So, let's take a look at the problem again. Here's the question.

- Find a matrix A so that the only solution of

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Because $A\mathbf{x}$ is a linear combination of the columns of A, and

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

we know that A has 3 entries in each column (3 rows). Therefore, A must look like this.

$$A = \begin{pmatrix} x & x & x & x & x & \cdots \\ x & x & x & x & x & \cdots \\ x & x & x & x & x & \cdots \end{pmatrix}$$

However, we're told that $\mathbf{x} = (0, 1)^T$ is a solution. Hence,

$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

This tells us that A must have two columns; zero multiplies the first column of A, 1 multiplies the second column. Thus, A must look like this:

$$A = \begin{pmatrix} x & x \\ x & x \\ x & x \end{pmatrix}.$$

Now, as I said in class today, one obvious idea is

$$\begin{pmatrix} x & 1 \\ x & 2 \\ x & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

After this I went a little cuckoo. Lee tried to help. So did Dave. But a senior moment is a senior moment. Oh well.

Now, matrix

$$A = \begin{pmatrix} x & 1 \\ x & 2 \\ x & 3 \end{pmatrix}$$

either has rank 1 or rank 2. If matrix A has rank 1, what happens? Here is an example of a rank 1 matrix, where column 1 is a multiple of column 2.

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Setting up the augmented matrix and reducing,

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

I see that this system has many solution. This will always be the case if I choose a rank 1 matrix A with $(1, 2, 3)^T$ as my second column. But Strang asks for a matrix A so that $\mathbf{x} = (0, 1)^T$ is the only solution of

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

So, I cannot choose a rank 1 matrix with second column $(1, 2, 3)^T$.

What happens if the rank is 2? Then the first column of A is independent of the second and A must reduce to

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, the system will have no solutions or a unique solution, depending on the value of $\mathbf{b}$. But I chose column 2 of matrix A so that at least one solution exists. Thus, I should be able to choose any vector independent of $(1, 3)^T$, stick it in the first column and be good to go.

So, choose a first column, making sure it is not a multiple of $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$, my second column.

$$\begin{pmatrix} 5 & 1 \\ 12 & 2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Set up the augmented matrix and reduce.

$$\begin{pmatrix} 5 & 1 & 1 \\ 12 & 2 & 2 \\ -3 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $x_1 = 0$ and $x_2 = 1$ is the only solution of this system. Ta Da!