

The trick here is to back up and first try problem 13b on page 162. I look at only one space. The column space of  $A$  is the same as the column space of  $-A$ . Why?

Let  $\mathbf{v}$  be an element of the column space of  $A$ . Then  $\mathbf{v}$  is a linear combination of the columns of  $A$ . Let

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_n].$$

Of course, this means that

$$-A = [-\mathbf{a}_1 \ -\mathbf{a}_2 \ \dots \ -\mathbf{a}_n].$$

Because  $\mathbf{v}$  is in the column space of  $A$ , we can write

$$\mathbf{v} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n.$$

However, this can be manipulated so that  $\mathbf{v}$  is in the column space of  $-A$ .

$$\begin{aligned} \mathbf{v} &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n \\ &= (-x_1)(-\mathbf{a}_1) + (-x_2)(-\mathbf{a}_2) + \dots + (-x_n)(-\mathbf{a}_n) \end{aligned}$$

Thus,  $\mathbf{v}$  is a linear combination of the columns of  $-A$  and is therefore in the column space of  $-A$ . Therefore, everything that lies in the column space of  $A$  also lies in the column space of  $-A$ . That is,  $C(A) \subset C(-A)$ .

In a similar manner, show that the column space of  $-A$  is contained in the column space of  $A$ . Start with an arbitrary linear combination of the columns of  $-A$  and show that it is also a linear combination of the columns of  $A$ . Then, you have  $C(-A) \subset C(A)$ .

The conclusion of these two ideas is the fact that  $C(A) = C(-A)$ .

With this exercise 13 completed, exercise 25d is a no-brainer. If  $A^T = -A$ , then

$$C(A^T) = C(-A) = C(A).$$

Done.