

# 1 The Dot Product

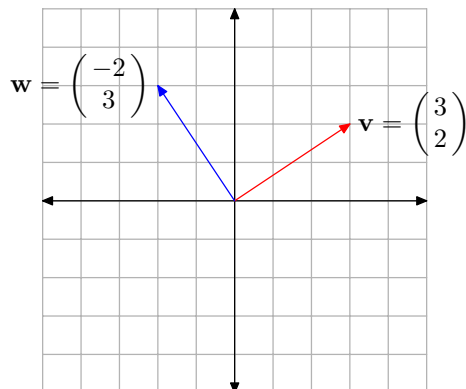
OK. Here's a little preview. Not a full explanation, just a little taste before the main meal. On your current homework, you are studying the dot product. If you have two vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix},$$

then the dot product of the two vectors is defined to be

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \mathbf{v}^T \mathbf{w} \\ &= (v_1 \quad v_2 \quad \dots \quad v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \\ &= v_1 w_1 + v_2 w_2 + \dots + v_n w_n. \end{aligned}$$

It's not hard to prove (see Strang, page 13) that the vectors are orthogonal (read perpendicular) when their dot product is zero. Indeed, consider the following example.



The vectors

$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

are clearly perpendicular.<sup>1</sup> Note that their dot product is

$$\mathbf{v} \cdot \mathbf{w} = (3)(-2) + (2)(3) = 0.$$

# 2 The Four Fundamental Spaces

Consider, if you will, the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 4 \\ 3 & 1 & 4 & 8 \end{pmatrix}.$$

We begin our discussion with a review of the four fundamental spaces involving the matrix  $A$ .

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<sup>1</sup>The product of their slopes is  $-1$ .

## 2.1 The Column Space of $A$

Reduce matrix  $A$ .

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 4 \\ 3 & 1 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 12/5 \\ 0 & 1 & 1 & 4/5 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

Columns 1 and 2 of matrix  $R$  are pivot columns. Therefore, columns 1 and 2 of matrix  $A$  form a basis for the column space of matrix  $A$ .

$$C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Note that dimension of the column space is 2. Note that each vector in the column space is a vector in  $\mathbf{R}^3$ .

## 2.2 The Null Space of $A$

The free columns of matrix  $R$  are columns 3 and 4. First, column 3 can be written as a linear combination of the pivot columns that proceed it. Indeed,

$$-1 \cdot \text{col } 1 - 1 \cdot \text{col } 2 + 1 \cdot \text{col } 3 = \mathbf{0}.$$

Consequently,

$$-1 \cdot \text{col } 1 - 1 \cdot \text{col } 2 + 1 \cdot \text{col } 3 + 0 \cdot \text{col } 4 = \mathbf{0}$$

and

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

is a vector in the nullspace of matrix  $R$ , and therefore also of  $A$ . Next, note column 4. Column 4 can be written as a linear combination of the pivot columns that proceed it. Indeed,

$$-(12/5) \cdot \text{col } 1 - (4/5) \cdot \text{col } 2 + 1 \cdot \text{col } 4 = \mathbf{0}.$$

Consequently,

$$-(12/5) \cdot \text{col } 1 - (4/5) \cdot \text{col } 2 + 0 \cdot \text{col } 3 + 1 \cdot \text{col } 4 = \mathbf{0}$$

and

$$\begin{pmatrix} -12/5 \\ -4/5 \\ 0 \\ 1 \end{pmatrix}$$

is a vector in the nullspace of matrix  $R$ , and therefore also of  $A$ . However, if  $A\mathbf{x} = \mathbf{0}$ , then

$$A(5\mathbf{x}) = 5(A\mathbf{x}) = 5\mathbf{0} = \mathbf{0},$$

and this means that the vector

$$\begin{pmatrix} -12 \\ -4 \\ 0 \\ 5 \end{pmatrix}$$

is also in the nullspace of matrix  $A$ . Consequently, a basis for the nullspace of  $A$  is made up of these “special” vectors.

$$N(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -12 \\ -4 \\ 0 \\ 5 \end{pmatrix} \right\}$$

Note that the dimension of the nullspace is 2. Further, note that each vector in the nullspace of  $A$  is a vector in  $\mathbf{R}^4$ .

## 2.3 The Row Space of $A$

The row space of matrix  $A$  is the set of all possible linear combinations of the rows of  $A$ . In practice, we say “row space,” but what we are really talking about is the column space of  $A^T$ . After all, the columns of  $A^T$  are the rows of  $A$ , just as the rows of  $A$  are the columns of  $A^T$ . Well, not really. After all, rows are rows, columns are columns, but I think you know what we intend. When we talk about the row space of matrix  $A$ , we’re really talking about the column space of  $A^T$ .

So, reduce  $A^T$ .

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \\ 4 & 4 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R$$

Note that column 1 and column 2 are pivot columns of  $R$ . Thus, column 1 and column 2 of matrix  $A^T$  form a basis for the column space of  $A^T$ .

$$C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 4 \end{pmatrix} \right\}$$

Note that the dimension of the column space of  $A^T$  (the row space of  $A$ ) is 2. Note further that the elements of the column space of  $A^T$  are vectors in  $\mathbf{R}^4$ .

## 2.4 The Null Space of $A^T$

Note that column 3 of matrix  $R$  above is free. Further, note that this column can be written as a linear combination of the pivot columns that proceed it. Indeed,

$$-1 \cdot \text{col } 1 - 1 \cdot \text{col } 2 + 1 \cdot \text{col } 3 = \mathbf{0}.$$

Therefore,

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

is a vector in the nullspace of  $R$ , and consequently of  $A^T$ . Thus, a basis for the nullspace of  $A^T$  is

$$N(A^T) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Note that the dimension of the nullspace of  $A^T$  (the row space of  $A$ ) is 1. Further, note that each element in the nullspace of  $A^T$  is a vector in  $\mathbf{R}^3$ .

## 2.5 The Dimensions Are Right

First, look at the row space and null space of matrix  $A$ .

$$C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 4 \end{pmatrix} \right\} \quad \text{and} \quad N(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -12 \\ -4 \\ 0 \\ 5 \end{pmatrix} \right\}$$

Note that these subspaces consist of vectors in  $\mathbf{R}^4$ . The row and nullspace of matrix  $A$  are subspaces of  $\mathbf{R}^4$ . Further, note that the dimensions are right. The dimension of the row space is 2. The dimension of the nullspace is 2. These dimensions add to 4, the dimension of the vector space  $\mathbf{R}^4$ . This is no coincidence. This is guaranteed to happen every time.

Now, look at the column space of  $A$  and the nullspace of  $A^T$ .

$$C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad N(A^T) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Note that these subspaces consist of vectors from  $\mathbf{R}^3$ . The column space of  $A$  and the nullspace of  $A^T$  are subspaces of  $\mathbf{R}^3$ . Further, note that the dimensions are right. The dimension of the column space of  $A$  is 2. The dimension of the nullspace of  $A^T$  is 1. The dimensions add to 3, the dimension of the vector space  $\mathbf{R}^3$ . Again, this is no coincidence. This is guaranteed to happen every time.

## 2.6 Orthogonality

Let's look at the row space of  $A$  and the nullspace of  $A$  once again.

$$C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 4 \end{pmatrix} \right\} \quad \text{and} \quad N(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -12 \\ -4 \\ 0 \\ 5 \end{pmatrix} \right\}$$

Note that the basis vectors for the row space and the nullspace of  $A$  are orthogonal. Indeed, if we let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} -12 \\ -4 \\ 0 \\ 5 \end{pmatrix},$$

then it is easy to show that

$$\mathbf{v}_1 \cdot \mathbf{w}_1 = 0, \quad \mathbf{v}_1 \cdot \mathbf{w}_2 = 0, \quad \mathbf{v}_2 \cdot \mathbf{w}_1 = 0, \quad \text{and} \quad \mathbf{v}_2 \cdot \mathbf{w}_2 = 0.$$

Indeed, we can go further. Take something from the column space of  $A^T$ . It can be written as a linear combination of the basis elements for the column space of  $A^T$ .

$$\mathbf{v} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2$$

Take something in the nullspace of  $A$ . It can be written as a linear combination of the basis elements for the nullspace of  $A$ .

$$\mathbf{w} = \gamma \mathbf{w}_1 + \delta \mathbf{w}_2$$

Now,

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) \cdot (\gamma \mathbf{w}_1 + \delta \mathbf{w}_2) \\ &= \alpha \gamma (\mathbf{v}_1 \cdot \mathbf{w}_1) + \alpha \delta (\mathbf{v}_1 \cdot \mathbf{w}_2) + \beta \gamma (\mathbf{v}_2 \cdot \mathbf{w}_1) + \beta \delta (\mathbf{v}_2 \cdot \mathbf{w}_2) \\ &= 0. \end{aligned}$$

Not only are the individual basis elements of the row space and nullspace orthogonal, but any element in the row space is orthogonal to any element in the nullspace. Moreover, all elements in  $\mathbf{R}^4$  are accounted for because the dimensions are right. We say that the nullspace of  $A$  is the *orthogonal complement* of the row space of  $A$ .

In similar fashion, you can show that elements in the column space of  $A$  are orthogonal to elements in the nullspace of  $A^T$ . Again the dimensions are right. All elements in  $\mathbf{R}^3$  are accounted for and we say that the nullspace of  $A^T$  is the orthogonal complement of the column space of  $A$ .