Strang
Exercise 12, Page 181

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We want to find a matrix that has

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\2\\0\end{bmatrix}$$

as a basis for its column space and

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1 & 2 & 0\end{bmatrix}$$

as a basis for its row space. Consider the product

$$\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix}.$$

Using block multiplication, this product is

$$\begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{bmatrix}.$$

The thing to note here is that each of the columns of this matrix are a multiple of the column vector

$$\begin{bmatrix}1\\0\\1\end{bmatrix},$$

and each of the rows of this matrix are a multiple of the row vector

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix}.$$

Similarly, consider the product

$$\begin{bmatrix}1\\2\\0\end{bmatrix}\begin{bmatrix}1 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 0\\2 & 4 & 0\\0 & 0 & 0\end{bmatrix}.$$

Each of the columns is a multiple of

$$\begin{bmatrix}1\\2\\0\end{bmatrix},$$

and each of the rows is a multiple of

$$\begin{bmatrix}1 & 2 & 0\end{bmatrix}.$$

Finally, let's add the two matrices.

$$\begin{aligned}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix}+\begin{bmatrix}1\\2\\0\end{bmatrix}\begin{bmatrix}1 & 2 & 0\end{bmatrix} &= \begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 2 & 0\\2 & 4 & 0\\0 & 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}2 & 2 & 1\\2 & 4 & 0\\1 & 0 & 1\end{bmatrix}.\end{aligned}$$

Note that the first column of this last result is found by adding the first columns of the two preceding matrices. However, in the first case, the first column is a multiple of $(1, 0, 1)^T$, and in the second case, the first column is a multiple of $(1, 2, 0)^T$. Thus, the first column of the last matrix is a linear combination of

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\2\\0\end{bmatrix}.$$

Similarly, the remaining columns of the last matrix are linear combination of these same two column vectors. Hence,

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\2\\0\end{bmatrix}$$

are a basis for the column space of

$$\begin{bmatrix}2 & 2 & 1\\2 & 4 & 0\\1 & 0 & 1\end{bmatrix}.$$

Let's look at that computation again.

$$\begin{aligned}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix}+\begin{bmatrix}1\\2\\0\end{bmatrix}\begin{bmatrix}1 & 2 & 0\end{bmatrix} &= \begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 2 & 0\\2 & 4 & 0\\0 & 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}2 & 2 & 1\\2 & 4 & 0\\1 & 0 & 1\end{bmatrix}.\end{aligned}$$

Note that the first row of the result is found by adding the first rows of the two preceding matrices. However, in the first case, the first row is a multiple of $(1, 0, 1)$, and in the second case, the first row is a multiple of $(1, 2, 0)$. Thus, the first row of the last matrix is a linear combination of

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1 & 2 & 0\end{bmatrix}.$$

Similarly, the remaining rows of the last matrix are linear combination of these same two row vectors. Hence,

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$$

are a basis for the row space of

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Now, Michael, you're concerned what happens when you reduce this matrix.

```
>> A=[2 2 1;2 4 0;1 0 1]

A =

     2     2     1
     2     4     0
     1     0     1

>> R=rref(A)

R =

    1.0000         0    1.0000
         0    1.0000   -0.5000
         0         0         0
```

There are pivots in the first and second columns of R, so that means that the first and second columns of the matrix A,

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

are a basis for the column space of A. And you are worried that the third column of A,

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

seems to be getting left out. But this is not a contradiction, as it is easy to show that each of the columns

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

can be written as linear combinations of the columns

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Similar remarks are in order for the basis of the row space.