You can cheat ahead and read Section 6.4, or you can peruse the following proof.

Theorem 1 *Suppose that matrix A is symmetric and has distinct eigenvalues λ_1 and λ_2. That is, $\lambda_1 \neq \lambda_2$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the eigenvectors associated with λ_1 and λ_2, respectively. That is,*

$$\begin{aligned} A\mathbf{x}_1 &= \lambda_1\mathbf{x}_1, \text{ and} \\ A\mathbf{x}_2 &= \lambda_2\mathbf{x}_2. \end{aligned}$$

Then, $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Proof. We know that $A\mathbf{x}_1 = \lambda\mathbf{x}_1$. Take the dot product of both sides of this equation with respect to $\mathbf{x}_2$.

$$\begin{aligned} A\mathbf{x}_1 &= \lambda_1\mathbf{x}_1 \\ (A\mathbf{x}_1)\cdot\mathbf{x}_2 &= (\lambda_1\mathbf{x}_1)\cdot\mathbf{x}_2 \\ (A\mathbf{x}_1)^T\mathbf{x}_2 &= (\lambda_1\mathbf{x}_1)^T\mathbf{x}_2 \end{aligned}$$

Now, $(A\mathbf{x}_1)^T = \mathbf{x}_1^TA^T$ and $(\lambda_1\mathbf{x}_1)^T = \lambda_1\mathbf{x}_1^T$, so we can write

$$\mathbf{x}_1^TA^T\mathbf{x}_2 = \lambda_1\mathbf{x}_1^T\mathbf{x}_2.$$

But A is symmetric, so $A^T = A$.

$$\mathbf{x}_1^TA\mathbf{x}_2 = \lambda_1\mathbf{x}_1^T\mathbf{x}_2.$$

Now we use the fact that $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$ to write

$$\begin{aligned} \mathbf{x}_1^T\lambda_2\mathbf{x}_2 &= \lambda_1\mathbf{x}_1^T\mathbf{x}_2 \\ \lambda_2\mathbf{x}_1^T\mathbf{x}_2 &= \lambda_1\mathbf{x}_1^T\mathbf{x}_2. \end{aligned}$$

Set one side equal to zero and factor.

$$\begin{aligned} \lambda_2\mathbf{x}_1^T\mathbf{x}_2 - \lambda_1\mathbf{x}_1^T\mathbf{x}_2 &= 0 \\ (\lambda_2 - \lambda_1)\mathbf{x}_1^T\mathbf{x}_2 &= 0 \end{aligned}$$

Thus, we have the product of two numbers, $\lambda_1 - \lambda_1$ and $\mathbf{x}_1^T\mathbf{x}_2$ equaling zero. But λ_1 and λ_2 are distinct; i.e., $\lambda_1 \neq \lambda_2$. Therefore, $\lambda_2 - \lambda_1 \neq 0$,

$$\mathbf{x}_1^T\mathbf{x}_2 = 0$$

making $\mathbf{x}_1$ and $\mathbf{x}_2$ orthogonal.

Is this an awesome proof or what?