

The transformation

$$T(v_1, v_2) = (v_1, v_1)$$

is easily shown to be a linear transformation.

1. First, we must show that $T(c\mathbf{v}) = cT(\mathbf{v})$, for all $\mathbf{v} \in R^2$ and all $c \in R$.

$$\begin{aligned} T(c\mathbf{v}) &= T\left(c \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} cv_1 \\ cv_1 \end{pmatrix} \\ &= c \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} \\ &= cT\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\ &= cT(\mathbf{v}) \end{aligned}$$

2. Secondly, we must show that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $u, v \in R^2$.

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} u_1 + v_1 \\ u_1 + v_1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} \\ &= T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + T\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

Thus, T is a linear transformation.

The *kernel* of T is the set of all $\mathbf{u} \in R^2$ such that $T(\mathbf{u}) = \mathbf{0}$. Thus,

$$\begin{aligned} T(\mathbf{u}) &= \mathbf{0} \\ T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ u_1 &= 0. \end{aligned}$$

Thus, the kernel of T is the set of all (u_1, u_2) such that $u_1 = 0$. In a word, this is all points on the vertical axis.

The *range* of T is the set of all $T(\mathbf{u})$, such that \mathbf{u} is in the domain of T . But,

$$T(\mathbf{u}) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix},$$

where u_1 is any real number. Geometrically, this is the set of all points that have equal x - and y -coordinates, that is, the points that lie on the line $y = x$ (or on the line $u_1 = u_2$, if you prefer).

Now, let's find the *matrix of transformation*. We do this by determining what T does to each of the standard basis elements for R^2 . Clearly,

$$T(\mathbf{e}_1) = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ and}$$
$$T(\mathbf{e}_2) = T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, the matrix of transformation is

$$A = [L(\mathbf{e}_1) \ L(\mathbf{e}_2)] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

That this is the correct matrix of transformation is easily checked.

$$T(\mathbf{v}) = A\mathbf{v} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix}$$

Now, the kernel of T is the set of all \mathbf{u} such that

$$T(\mathbf{u}) = \mathbf{0}$$
$$A\mathbf{u} = \mathbf{0}.$$

Therefore, the kernel of T now becomes the nullspace of the matrix of transformation. Of course, the nullspace of

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is easily seen to be all multiples of the vector $(0, 1)^T$, which of course, is a geometric interpretation of the vertical axis, exactly what we got on the previous page.

The range of T is the set of all

$$T(\mathbf{u}) = A\mathbf{u},$$

such that \mathbf{u} is in the domain of T . Of course, $A\mathbf{u}$ is an element of the column space of A . Thus, the range of T should be the column space of matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

which, due to the single pivot column, is all multiples of the first column of A , $(1, 1)^T$. Of course, this is the line $y = x$ in the plane.

This stuff is tight!