You really have to spend quite a bit of time thinking about matrix multiplication in all of its forms to be successful. These concepts are not learned overnight, but only with experience over the long haul, so don't be frustrated when they don't come immediately. I have the same difficulties with matrix multiplication after years of teaching linear algebra.

If A is 3×5, as in

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

then the matrix-vector product $A\mathbf{x}$ is valid only if $\mathbf{x}$ has five entries, as in

$$\mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Remember, when multiplying a matrix times a vector, we take a linear combination of the columns of the matrix, using the components of the vector as weights. In this case,

$$\begin{aligned} A\mathbf{x} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \\ &= 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ -9 \\ -12 \end{pmatrix}. \end{aligned}$$

This is easily verified in Matlab. Enter the matrix A and the vector $\mathbf{x}$.

```
>> A=[1 2 3 4 5;2 3 4 5 6;3 4 5 6 7]
A =
     1     2     3     4     5
     2     3     4     5     6
     3     4     5     6     7
>> x=[1;-3;-2;0;1]
x =
     1
    -3
    -2
     0
     1
```

Now form the matrix-vector product.

```
>> A*x
ans =
    -6
    -9
   -12
```

Thus, if matrix A has 5 columns, then vector $\mathbf{x}$ must have 5 rows, because the matrix-vector product requires that we take a linear combination of the columns of matrix A, and this requires 5 scalars, one for each column.

Secondly, a linear combination of columns vectors, each having 3 components, yields a column vector having 3 components.

Next we have a form of matrix-matrix multiplication that is explained in terms of blocks. If matrix A and B have dimensions that make multiplication possible, maybe $m \times n$ and $n \times p$, and $\mathbf{b}_1$, $\mathbf{b}_2$, …, $\mathbf{b}_p$ represent the columns of matrix B, than AB is defined as

$$AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_p] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \ldots\ A\mathbf{b}_p].$$

That is, the first column of matrix AB is found by multiplying A times the first column of B, the second column of matrix AB is found by multiplying A times the second column of B, etc.

This is also easily verified in Matlab. As an example, let's form two columns for matrix B.

```
>> b1=x
b1 =
     1
    -3
    -2
     0
     1

>> b2=[1;-1;-2;-3;4]
b2 =
     1
    -1
    -2
    -3
     4
```

Let's compute $A\mathbf{b}_1$ and $A\mathbf{b}_2$.

```
>> A*b1
ans =
    -6
    -9
   -12

>> A*b2
ans =
     1
     0
    -1
```

Craft the matrix $B = [\mathbf{b}_1\ \mathbf{b}_2]$.

```
>> B=[b1,b2]
B =
     1     1
    -3    -1
    -2    -2
     0    -3
     1     4
```

Now find the matrix product AB.

```
>> A*B
ans =
    -6     1
    -9     0
   -12    -1
```

Note that the first column of AB is $A\mathbf{b}_1$ and the second column of AB is $A\mathbf{b}_2$, just as predicted by the theory.

In the discussion above we established that the matrix-vector product $A\mathbf{b}_j$ is defined only if the vector $\mathbf{b}_j$ has the same number of rows as A has columns. Because $B = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{v}_p]$, matrix B must have the same number of rows as A has columns. If A is $m \times n$, A has n columns. Therefore, B must have n rows, so its dimensions must be $n \times p$.

Secondly, if matrix A is $m \times n$, that means that each of its n columns has m rows. Thus, each matrix-vector product $A\mathbf{b}_j$, being a linear combination of the columns of A, must have m rows. Since B has p columns, this means that matrix AB must contain m rows and p columns.

Now to your question. You asked how to find the entry in row 3, column 4 of matrix AB. In your example, matrix A has dimensions 3×5 and matrix B has dimensions 5×3. Thus, the matrix B has the same number of rows as matrix A has columns. This means that the multiplication is possible. Also, matrix AB would have dimensions 3×3. Finally, as you correctly said, matrix AB would have **no** entry is row 3, column 4. All of this is correct.

However, there is no mention of the size of matrices A and B in the instructions of the exercise in Strang. So, if the matrices are big enough, and the dimensions are such that the multiplication is possible, what would be the entry in the row 3, column 4 of AB, provided such an entry exists? How is it found? I believe that is the intent of this question.

Of course, the entry in row 3, column 4 of matrix AB is found by taking the dot product of the third row of matrix A with the fourth column of matrix B.