Strang, Page 68, Exercise #23(c)

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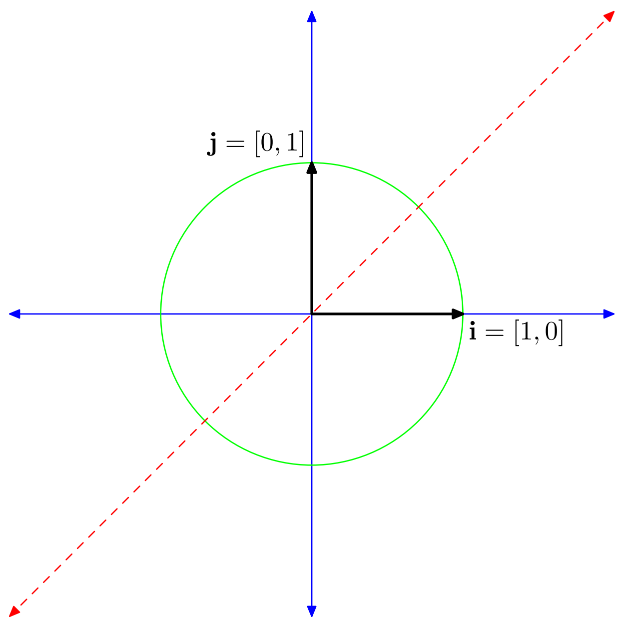
I began thinking about this exercise (find D and E so that $DE = -ED$) when I got home. I wondered if thinking about the geometry might help, so I drew a picture of the unit circle, the vector $\mathbf{i} = [1, 0]$ and $\mathbf{j} = [0, 1]$, and the line $y = x$.



I want to find two geometric transformations (such as scaling, projecting, reflecting, rotating) so that when the order of application to any vector is reversed, the resulting vector points in a direction opposite to the first ordering. Reflections came to mind. In particular:

1. Let D be the operation that reflects across the line $y = x$.

2. Let E be the operation that reflects across the x-axis.

We've seen that any transformation is completed determined by what it does to the unit vectors $\mathbf{i}$ and $\mathbf{j}$. First, let's see what ED does to the vector $\mathbf{i}$.

- Apply D, which reflects $\mathbf{i}$ across the line $y = x$ to the vector $\mathbf{j}$.

- Apply E to the result, which reflects the vector $\mathbf{j}$ across the x-axis to the vector $-\mathbf{j}$.
- Hence, $ED\mathbf{i} = -\mathbf{j}$.

Now, what happens when we apply DE to the vector $\mathbf{i}$.

- Apply E, which reflects $\mathbf{i}$ across the x-axis. Hence $\mathbf{i}$ remains fixed.
- Apply D to the result, which reflects $\mathbf{i}$ across the line $y = x$ to the vector $\mathbf{j}$.
- Hence, $DE\mathbf{i} = \mathbf{j}$, which is the negative of $ED\mathbf{i}$ above.

Hmmmm. Promising. If only the same thing will happen to the other unit vector $\mathbf{j}$. First, let's see what ED does to the vector $\mathbf{j}$.

- Apply D, which reflects $\mathbf{j}$ across the line $y = x$ to the vector $\mathbf{i}$.
- Apply E to the result, which reflects $\mathbf{i}$ across the x-axis. Hence, $\mathbf{i}$ remains fixed.
- Hence, $ED\mathbf{j} = \mathbf{i}$.

Now, let's see what happens when we apply DE to the vector $\mathbf{j}$.

- Apply E, which reflect $\mathbf{j}$ across the x-axis to $-\mathbf{j}$.
- Apply D to the result, which reflects $-\mathbf{j}$ across the line $y = x$ to $-\mathbf{i}$.
- Hence, $DE\mathbf{j} = -\mathbf{i}$, which is the negative of $ED\mathbf{j}$ above.

Hence, and DE and ED should transform a vector into vectors that point directly opposite to one another. That is,

$$DE\mathbf{x} = -ED\mathbf{x},$$

for any vector $\mathbf{x}$. Hence, DE should equal $-ED$. Let's check.

1. The matrix D that will reflect a vector across the line $y = x$ is the matrix

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2. The matrix E that will reflect a vector across the x-axis is the matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now,

$$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

On the other hand,

$$ED = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that

$$DE = -ED.$$