First, you have to read your text and discover why the following is true. If A and B are invertible, then so is AB and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Note the reversal of order. After all, if I put on my socks and then my shoes, the inverse of that is to remove my shoes first, then my socks. So, the above ordering is reasonable. I'll have more to say about this next class.

OK, so let's accept that $(AB)^{-1} = B^{-1}A^{-1}$ and proceed. But how? Shall we take the hint from my last note and play with some numbers again? That might give us a feel for the problem.

Hmmm, we know how to take the determinant of a 2×2 matrix. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

OK, 2×2 matrices are easy to invert, so let's stick with those. Let's fire up an example with numbers. Maybe that will give us a way to proceed. If

$$A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{(3)(2) - (1)(5)} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}.$$

Now, Strang wants to know what happens when we switch a pair of rows, but I am going to switch a pair of columns so that I don't completely give away the solution. That said, let

$$B = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}.$$

The inverse of B is

$$B^{-1} = \frac{1}{(1)(5) - (2)(3)} \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}.$$

Goodness! If I switch the columns of A the resulting matrix has an inverse which is identical to the inverse of A, only it has its rows switched! Could this be true in general? (*Here I would encourage you to try a 3×3 example with numbers and see if our conjecture still holds.*)

Yikes! I understand the numerical example, but how can I prove this in general. Think, David! How does one switch two columns. Ah! Elementary matrices. If I take

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and switch columns 1 and 2, then I get the elementary matrix

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And, on top of that, I saw today that if I multiply matrix A *on the right* by this elementary matrix, it should switch the same two columns of A!

$$AE = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = B$$

Aha! Matrix B equals AE! Now, remember our property above? $(AB)^{-1} = B^{-1}A^{-1}$. Therefore,

$$B^{-1} = (AE)^{-1} = E^{-1}A^{-1}.$$

But what is the inverse of E? Hmmm. Matrix E switches columns 1 and 2, so the inverse of that would be to switch columns 1 and 2 again. That way we'd be back to the starting matrix. Ah! The inverse of matrix E is itself. That is,

$$E^{-1} = E.$$

Therefore,

$$B = EA^{-1}.$$

I now know that I build B out of A^{-1} by multiplying it by the matrix E on the *left*! But, if I multiply on the *left* by matrix E,

$$B^{-1} = EA^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix},$$

so matrix B^{-1} is built by switching rows 1 and 2 of matrix A^{-1}, exactly as predicted by my play above!

Yippee!