If we multiply on the left by a permutation matrix, we will exchange some rows. If we multiply on the left by *another* permutation matrix, we will exchange some more rows. Therefore, the product of these two permutation matrices switch a bunch of rows. Therefore, the product of two (or more) permutation matrices is a permutation matrix.

Some times permutation matrices commute, sometimes they don't. For example, let P_{13} switch rows 1 and 3. Let P_{23} switch rows 2 and 3. Think 3×3.

Now, if I start with vector

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

and I hit it with $P_{23}P_{13}$, then

$$P_{23}P_{13}\mathbf{x} = P_{23}P_{13}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = P_{23}\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}.$$

However, if I hit the vector with $P_{13}P_{23}$, then

$$P_{13}P_{23}\mathbf{x} = P_{13}P_{23}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = P_{13}\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix}.$$

That's different! So the matrices won't commute! But you should use matrix multiplication to show that

$$P_{23}P_{13} \neq P_{13}P_{23}.$$

Now, what sort of thinking can you use to get a pair of permutation matrices that do commute?