Solution to Problem 4(e) in Section 3.6

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Abstract

This article proves that if the column space and row space of a matrix are equal, then the left nullspace and the nullspace are also equal.

1. Introduction

We want to show that if $C(A) = C(A^T)$, then $N(A) = N(A^T)$. In order to prove this we must show:

1. $N(A) \subset N(A^T)$
2. $N(A^T) \subset N(A)$

If we let $\mathbf{x} \in N(A)$ and then show that $\mathbf{x} \in N(A^T)$, it logically follows that $N(A) \subset N(A^T)$. In other words we want to show that each vector $\mathbf{x}$ in $N(A)$ will also be in $N(A^T)$.

If we also let $\mathbf{x} \in N(A^T)$ and then show that $\mathbf{x} \in N(A)$, it follows that $N(A^T) \subset N(A)$.

Then we will have proven that $N(A) = N(A^T)$.

2. Part 1

Show: $N(A) \subset N(A^T)$

$A\mathbf{y}$ be some linear combination of the columns of A and let $A^T\mathbf{y}$ be some linear combination of the rows of A (columns of A^T). Since $C(A) = C(A^T)$, there is a $\mathbf{y^*}$ such that $A\mathbf{y} = A\mathbf{y^*}$.

$\mathbf{x} \in N(A)$

$$\begin{aligned} \mathbf{x}^T(A\mathbf{y}) &= \mathbf{x}^T(A^T\mathbf{y^*}) \\ &= (\mathbf{x}^T A^T)\mathbf{y^*} \\ &= (A\mathbf{x})^T\mathbf{y^*} \\ &= \mathbf{0}^T\mathbf{y^*} \\ &= 0 \end{aligned} \tag{1}$$

ch column in A is (no surprise) a linear combination of the columns of A. For example e first column in A is equal to 1 × column 1 of itself.

$$\begin{aligned} \mathbf{x}^T A &= \mathbf{x}^T \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} \end{aligned} \tag{2}$$

Because $\mathbf{x}^T A = \mathbf{0}$

$$\begin{aligned} (\mathbf{x}^T A)^T &= \mathbf{0}^T \\ A^T(\mathbf{x}^T)^T &= \mathbf{0} \\ A^T\mathbf{x} &= \mathbf{0} \end{aligned} \tag{3}$$

Therefore $\mathbf{x} \in N(A^T)$.

Since we let $\mathbf{x} \in N(A)$ to begin with, we have proven

$$N(A^T) \subset N(A)$$

3. Part 2

Now we must show that the opposite is true.

$$N(A) \subset N(A^T)$$

Let $\mathbf{x} \in N(A^T)$

$$\begin{aligned} \mathbf{x}^T(A^T\mathbf{y}) &= \mathbf{x}^T(A\mathbf{y}^*) \\ &= (\mathbf{x}^T A)\mathbf{y}^* \\ &= (A^T\mathbf{x})^T\mathbf{y}^* \\ &= \mathbf{0}^T\mathbf{y}^* \\ &= 0 \end{aligned} \tag{4}$$

Using the same reasoning as in Part 1:

$$\begin{aligned} \mathbf{x}^T A^T &= \mathbf{x}^T \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_n \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{5}$$

Now because $\mathbf{x}^T A^T = \mathbf{0}$

$$\begin{aligned} (\mathbf{x}^T A^T)^T &= \mathbf{0}^T \\ A\mathbf{x} &= \mathbf{0} \end{aligned} \tag{6}$$

Therefore $\mathbf{x} \in N(A)$

Since we let $\mathbf{x} \in N(A^T)$ to begin with, then we have proven:

$$N(A) \subset N(A^T)$$

4. Conclusion

Since we have proven $N(A^T) \subset N(A)$ and $N(A) \subset N(A^T)$ we can now say that:

$$N(A) = N(A^T)$$

In simpler terms, because every element of $N(A^T)$ is in N(A), and every element of $N(A)$ is in $N(A^T)$, these spaces must be equal.  lause.

5. Bibliography, References, and Credits

David Arnold. 2001  The Source Of All That Is Math: A Contemporary Perspective.

Doug Saucedo. Special Thanks for Fetching Scratch Paper.

Gilbert Strang. 2001 *Introduction To Linear Algebra*