Solution to Problem 4(e) in Section 3.6

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Abstract

This article proves that if the column space and row space of a matrix are equal, then the left nullspace and the nullspace are also equal.

1. How to Show Two Sets are Equal

We want to show that if $C(A) = C(A^T)$, then $N(A) = N(A^T)$. In order to prove this we must show:

1. $N(A) \subset N(A^T)$
2. $N(A^T) \subset N(A)$

If we let $\mathbf{x} \in N(A)$ and then show that $\mathbf{x} \in N(A^T)$, it logically follows that $N(A) \subset N(A^T)$. In other words we want to show that each vector $\mathbf{x}$ in $N(A)$ will also be in $N(A^T)$.

If we also let $\mathbf{x} \in N(A^T)$ and then show that $\mathbf{x} \in N(A)$, it follows that $N(A^T) \subset N(A)$.

Then we will have proven that $N(A) = N(A^T)$.

2. Part 1

To understand our first proof that $N(A) \subset N(A^T)$, it is important to note that A$\mathbf{y}$ is a linear combination of the columns of A. Equally important is the fact that A$^T\mathbf{y}$ is a linear combination of the columns of A^T (rows of A). Since $C(A) = C(A^T)$, that is, since the column spaces of A and A^T are equal, for each $\mathbf{y}$ there is a $\mathbf{y*}$ such that $A\mathbf{y} = A\mathbf{y*}$.

Let $\mathbf{x} \in N($

$$\begin{aligned}\mathbf{x}^T(A\mathbf{y}) =& \mathbf{x}^T(A^T\mathbf{y*}) \\ =&(\mathbf{x}^T A^T)\mathbf{y*} \\ =&(A\mathbf{x})^T\mathbf{y*} \\ =&\mathbf{0}^T\mathbf{y*} \\ =& \end{aligned} \tag{1}$$

This result indicates that if we multiply $\mathbf{x}^T$ times any linear combination of the columns of A, the result is the number zero. Each column in A is (no surprise) a linear combination of the columns of A. For example the first column in A is equal to 1 × column 1 of itself. So if we multiply A by $\mathbf{x}^T$, we get the zero vector.

$$\begin{aligned}\mathbf{x}^T A &= \mathbf{x}^T \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \dots & \mathbf{a}_n\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{x}^T\mathbf{a}_1 & \mathbf{x}^T\mathbf{a}_2 & \mathbf{x}^T\mathbf{a}_3 & \dots\mathbf{x}^T\mathbf{a}_n\end{bmatrix} \\ &= \begin{bmatrix}0 & 0 & 0 & \dots & 0\end{bmatrix}\end{aligned} \tag{2}$$

If we take the transpose of the equation $\mathbf{x}^T A = \mathbf{0}$, we get

$$(\mathbf{x}^T A)^T = \mathbf{0}^T$$
$$A^T (\mathbf{x}^T)^T = \mathbf{0} \tag{3}$$
$$A^T \mathbf{x} = \mathbf{0}.$$

Therefore $\mathbf{x} \in N(A^T)$.

Since we let $\mathbf{x} \in N(A)$ to begin with, we have proven that

$$N(A^T) \subset N(A).$$

3. Part 2

Now we must show that the opposite is true.

$$N(A) \subset N(A^T)$$

Let $\mathbf{x} \in N(A^T$

$$\mathbf{x}^T (A^T \mathbf{y}) = \mathbf{x}^T (A\mathbf{y}*)$$
$$= (\mathbf{x}^T A)\mathbf{y}*$$
$$= (A^T \mathbf{x})^T \mathbf{y}* \tag{4}$$
$$= \mathbf{0}^T \mathbf{y}*$$
$$= ($$

Using the same reasoning as in Part 1:

$$\mathbf{x}^T A^T = \mathbf{x}^T \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

As before, if we take the transpose of the equation $\mathbf{x}^T A^T = \mathbf{0}$, we get

$$(\mathbf{x}^T A^T)^T = \mathbf{0}^T \tag{6}$$
$$A\mathbf{x} =$$

Therefore $\mathbf{x} \in N($

Since we let $\mathbf{x} \in N(A^T)$ to begin with, then we have prov

$$N(A) \subset N(A^T$$

4. Conclusion

Since we have proven $N(A^T) \subset N(A)$ and $N(A) \subset N(A^T)$ we can now say tha

$$N(A) = N(A^T$$

In simpler terms, because every element of $N(A^T)$ is in $N(A)$, and every element of $N(A)$ is in $N(A^T)$, these spaces must be equal.

5. Acknowledgements

David Arnold. 2001 The Source Of All That Is Math: A Contemporary Perspective.

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