Proof that if the Column Space and the Row Space of a Matrix are Equal, then the Nullspace and Left Nullspace are Equal

Greg Brown and Jonathan Gent

December 14, 2003

1 How to Show Two Sets are Equal

We want to show that if $C(A) = C(A^T)$, then $N(A) = N(A^T)$. In order to prove this, we must show:

1. $N(A) \subset N(A^T)$
2. $N(A^T) \subset N(A)$

If we let $\mathbf{x} \in N(A)$ and then show $\mathbf{x} \in N(A^T)$, it follows that $N(A) \subset N(A^T)$. In other words, we want to show that each vector $\mathbf{x}$ in $N(A)$ will also be in $N(A^T)$.

If we also let $\mathbf{x} \in N(A^T)$ and then show that $\mathbf{x} \in N(A)$, it follows that $N(A^T) \subset N(A)$. Then we will have proved that $N(A) = N(A^T)$.

2 Proof that $N(A) \subset N(A^T)$

To understand our first proof that $N(A) \subset N(A^T)$, it is important to note that $A\mathbf{y}$ is a linear combination of the columns of A. Equally important is the fact that $A^T\mathbf{y}$ is a linear combination of the columns of A^T (the rows of A). Since $C(A) = C(A^T)$, that is, since the column spaces of A and A^T are equal, for each $\mathbf{y}$ there is a $\mathbf{y}^*$ such that $A\mathbf{y} = A^T\mathbf{y}^*$. Let $\mathbf{x} \in N(A)$. Since $C(A) = C(A^T)$, the dimension of each column of A is equal to the dimension of each row of A. Therefore we can multiply A by $\mathbf{x}^T$, since $\mathbf{x}^T$ has the same dimension as each column in A. Then,

$$\begin{aligned}
\mathbf{x}^T(A\mathbf{y}) &= \mathbf{x}^T(A^T\mathbf{y}^*) \\
&= (\mathbf{x}^T A^T)\mathbf{y}^* \\
&= (A\mathbf{x})^T\mathbf{y}^* \\
&= \mathbf{0}^T\mathbf{y}^* \\
&= 0.
\end{aligned}$$

This result indicates that if we multiply $\mathbf{x}^T$ times any linear combination of the columns of A, the result is the scalar number zero. Each column in A is (no surprise) a linear combination of the columns of A. For example, the first column in A is equal to 1 multiplied by itself. So if we multiply A by $\mathbf{x}^T$, we get the zero vector:

$$\begin{aligned}\mathbf{x}^T A &= \mathbf{x}^T[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \ldots\ \mathbf{a}_n] \\ &= [\mathbf{x}^T\mathbf{a}_1\ \mathbf{x}^T\mathbf{a}_2\ \mathbf{x}^T\mathbf{a}_3\ \ldots\ \mathbf{x}^T\mathbf{a}_n] \\ &= [0\ 0\ 0\ \ldots\ 0] \\ &= \mathbf{0}.\end{aligned}$$

If we take the transpose of the equation $\mathbf{x}^T A = \mathbf{0}$, we get

$$\begin{aligned}(\mathbf{x}^T A)^T &= \mathbf{0}^T \\ A^T(\mathbf{x}^T)^T &= \mathbf{0} \\ A^T\mathbf{x} &= \mathbf{0}.\end{aligned}$$

Therefore $\mathbf{x} \in N(A^T)$. Since we let $\mathbf{x} \in N(A)$ to begin with, we have proved that

$$N(A) \subset N(A^T).$$

3 Proof that $N(A^T) \subset N(A)$

Now we must show that the opposite is true:

$$N(A^T) \subset N(A).$$

Let $\mathbf{x} \in N(A^T)$. Then,

$$\begin{aligned}\mathbf{x}^T(A^T\mathbf{y}^*) &= \mathbf{x}^T(A\mathbf{y}) \\ &= (\mathbf{x}^T A)\mathbf{y} \\ &= (A^T\mathbf{x})^T\mathbf{y} \\ &= \mathbf{0}^T\mathbf{y} \\ &= 0.\end{aligned}$$

Now let $[\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$ be the columns of A^T. Then, using similar reasoning as that used in our proof of $N(A) \subset N(A^T)$,

$$\begin{aligned}\mathbf{x}^T A^T &= \mathbf{x}^T \left[\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \ldots\ \mathbf{a}_n\ \right] \\ &= \left[\ \mathbf{x}^T\mathbf{a}_1\ \mathbf{x}^T\mathbf{a}_2\ \mathbf{x}^T\mathbf{a}_3\ \ldots\ \mathbf{x}^T\mathbf{a}_n\ \right] \\ &= \left[\ 0\ 0\ 0\ \ldots\ 0\ \right] \\ &= \mathbf{0}.\end{aligned}$$

As before, if we take the transpose of the equation $\mathbf{x}^T A^T = \mathbf{0}$, we get

$$\begin{aligned}(\mathbf{x}^T A^T)^T &= \mathbf{0}^T \\ A\mathbf{x} &= \mathbf{0}.\end{aligned}$$

Therefore $\mathbf{x} \in N(A)$. Since we let $\mathbf{x} \in N(A^T)$ to begin with, then we have proved

$$N(A^T) \subset N(A).$$

4 Conclusion

Since we have proven $N(A^T) \subset N(A)$ and $N(A) \subset N(A^T)$ we can now say that

$$N(A) = N(A^T).$$

In other words, because every element of $N(A^T)$ is in $N(A)$, and every element of $N(A)$ is in $N(A^T)$, these spaces must be equal.

5 Acknowledgements

David Arnold. 2001. The Source Of All That Is Math: A Contemporary Perspective.
Mr. Douglas Saucedo. Special Thanks For Fetching Scratch Paper.
Gilbert Strang. 2001. *Introduction To Linear Algebra.*